NASA Contractor Report 172416

ICASE REPORT NO. 84-28

ICASE

FAST SOLUTIONS TO THE STEADY STATE COMPRESSIBLE AND INCOMPRESSIBLE FLUID DYNAMIC EQUATIONS

Eli Turkel

Contract Nos. NAS1-17070, NAS1-17130

June 1984

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

FAST SOLUTIONS TO THE STEADY STATE COMPRESSIBLE

AND INCOMPRESSIBLE FLUID DYNAMIC EQUATIONS

Eli Turkel
Tel-Aviv University, Tel-Aviv, Israel
and
Institute for Computer Applications in Science and Engineering

Abstract

It is well known that for low speed flows the use of the compressible fluid dynamic equations is inefficient. The use of an explicit scheme requires Δt to be bounded by $1/c$. However, the physical parameters change over time scales of order $1/u$ which is much larger. Hence, it is not appropriate to use explicit schemes for very subsonic flows. Implicit schemes are hard to vectorize and frequently do not converge quickly for very subsonic flows. We shall demonstrate that if one is only interested in the steady state then a minor change to an existing code can greatly increase the efficiency of an explicit method. Even when using an implicit method the proposed changes increase the efficiency of the scheme. We shall first consider the Euler equations for low speed flows and then incompressible flows. We then indicate how to generalize the method to include viscous effects. We also show how to accelerate supersonic flow by essentially decoupling the equations.

Research was supported by the National Aeronautics and Space Adminstration under Contracts No. NAS1-17070 and NAS1-17130 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

Euler Equations for Subsonic Flow

We first consider low speed flows for rotational inviscid flow. Since the flow may be rotational we consider the Euler equations rather than the potential equation. We only consider schemes in conservation form. The use of conservation form allows the same code to be used for highly subsonic, transonic and supersonic flows.

The Euler equations, in two space dimensions, can be expressed as

$$w_t + f_x + g_y = 0 \tag{1}$$

where (x,y) represent general curvilinear coordinates. Since we are only interested in the steady state we replace (1) by the system

$$M^{-1} w_t + f_x + g_y = 0. \tag{2}$$

The requirements on M are that the matrix be nonsingular and that the resultant equations be well-posed. It is straightforward to solve (2) with an explicit scheme. With an implicit method only the diagonal portion of the matrix to be inverted is changed. Though the code solves (2) we shall only analyze the constant coefficient problem

$$M^{-1} w_t + Aw_x + Bw_y = 0 \tag{3}$$

where the matrices M, A, B are constant. Let $w^{(0)} = Tw$, $A_0 = T A T^{-1}$, $B_0 = T B T^{-1}$, $M_0^{-1} = T M^{-1} T^{-1}$, where T is chosen appropriately [6], then (3) can be converted to

$$M_0^{-1}\, w_t^{(0)} + A_0\, w_x^{(0)} + B_0\, w_y^{(0)} = 0, \tag{4}$$

with

$$A_0 = \begin{bmatrix} q & Y_y c & -X_y c & 0 \\ Y_y c & q & 0 & 0 \\ -X_y c & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \qquad B_0 = \begin{bmatrix} r & -Y_x c & X_x c & 0 \\ -Y_x c & r & 0 & 0 \\ X_x c & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix} \tag{5}$$

$$q = Y_y u - X_y v, \qquad r = X_x v - Y_x u \tag{6}$$

q and r are the contravariant components of the velocity and (X,Y) are the Cartesian coordinates.

We now consider the case of low speed flows. We wish to choose M_0^{-1} so that the eigenvalues of $M_0\, A_0$ and $M_0\, B_0$ are independent of c. We also wish to choose M_0 to be positive definite. This will imply that (4) is a symmetric hyperbolic system and so well-posed. One choice is

$$M_0^{-1} = \begin{bmatrix} \frac{c^2}{z^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

where $z^2 = \max(\varepsilon,\, u^2 + v^2)$. ε is introduced so that the matrix M is not singular at stagnation points. Transforming back to curvilinear coordinates we find that $M = I + dQ$, $d = (\gamma-1)(z^2/c^2-1)/c^2$

$$Q = \begin{bmatrix} s^2 & -u & -v & 1 \\ us^2 & -u^2 & -uv & u \\ vs^2 & -uv & -v^2 & v \\ hs^2 & -uh & -vh & h \end{bmatrix} \tag{8}$$

where

$$s^2 = (u^2 + v^2)/2, \quad h = c^2/(\gamma-1) + s^2. \tag{9}$$

We note that given the first row of Q the following rows are derived by multiplying the first row by u, v, h respectively. Hence the product of Q times a vector requires only six multiplications.

Let $\overline{M}^2 = z^2/c^2$ then the largest eigenvalue of $D = A \sin\theta + B \sin\phi$ is given by

$$\lambda = \frac{|w|(1 + \overline{M}^2) + \sqrt{w^2(1 - \overline{M}^2) + 4(a^2 + b^2)z^2}}{2} \tag{10}$$

where

$$w = q \sin\theta + r \sin\phi, \quad a = Y_y \sin\theta - Y_x \sin\phi, \quad b = X_x \sin\phi - X_y \sin\theta.$$

We see that near a stagnation point $M = O(\varepsilon)$, $\lambda = O(\varepsilon)$. While at $\overline{M} = 1$, $\lambda = |w| + \sqrt{a^2 + b^2}\, c$. Hence, at small Mach numbers the largest eigenvalue, and hence the time step, is independent of the sound speed c. At transonic sound speeds the largest eigenvalue is comparable to the nonconditioned case. Hence, the preconditioned problem allows a larger time step for all subsonic flows (see also [5] for the isoenergetic case).

We next consider incompressible flow. In conservation form the system is given by

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + (u^2 + p)_x + (uv)_y &= 0 \\ v_t + (uv)_x + (v^2 + p)_y &= 0 \end{aligned} \tag{11}$$

where p is the pressure normalized by the density. We wish to integrate this system using the artificial compressibility method [1]. If p_t/c^2 is inserted in the first equation the system is hyperbolic but not well conditioned [2]. Instead we replace (11) by

$$\begin{aligned} \frac{1}{c^2} p_t + u_x + v_y &= 0 \\ \frac{u}{c^2} p_t + u_t + (u^2 + p)_x + (uv)_y &= 0 \\ \frac{v}{c^2} p_t + v_t + (uv)_x + (v^2 + p)_y &= 0, \end{aligned} \tag{12}$$

c is a given nonzero function. It is evident that at the steady state both systems coincide. The new system can readily be shown to be unitarily equivalent to a symmetric hyperbolic system and so is well-posed and well-conditioned. The extra time derivatives can be considered as a preconditioning matrix similar to that previously considered. It remains to decide how to choose the function c. For c larger the coefficient of many time derivatives is small, however the time step of an explicit scheme will be large.

Hence, in this case c is merely a scaling of the time scale. Instead, we wish to choose c as large as possible without the need for decreasing in any substantial way the stability criterion. In one space dimension the stability criterion for (12) is given by

$$\frac{\Delta t}{\Delta x} < \frac{u + \sqrt{u^2 + 4c^2}}{2} . \qquad (13)$$

Hence, a reasonable choice for c is

$$c^2 = \max\left(\frac{u^2 + v^2}{4} , \varepsilon^2\right). \qquad (14)$$

Viscous Flow

We next consider the incompressible steady state Navier-Stokes equation

$$\begin{aligned} u_x + v_y &= 0 \\ (u^2 + p)_x + (uv)_y &= \frac{1}{R}\left(u_{xx} + u_{yy}\right) \qquad (15) \\ (uv)_x + (v^2 + p)_y &= \frac{1}{R}\left(v_{xx} + v_{yy}\right). \end{aligned}$$

As before we shall consider a pseudo time dependent approach to the steady state. We consider two ways of extending the previous results to include viscous effects.

The first possibility is to use a leapfrog method for the inviscid part and a Dufort-Frankel scheme for the viscous portion [1]. The inviscid part

bounds the time step by $\Delta x/\sqrt{u^2 + v^2}$ while the viscous part is unconditionally stable. The second possibility is to use a semi-implicit method based on the preconditioned system. The advection terms are treated explicitly while the viscous terms are treated implicitly. As before, the explicit part restricts the time steps inversely proportional to the velocity. The implicit part only contains linear terms. Hence, the implicit part can be inverted without any need for an iterative method.

Supersonic Flow

We next consider supersonic inviscid flow. In this case the matrices A and B can be simultaneously diagonalized by a congruence transform [4]. Hence we can find a matrix M so that (2) is similar to a diagonal system (for the linearized constant coefficient problem). Thus, in the supersonic regime we can choose a different time step for each equation. The resultant system is still hyperbolic and well-posed.

References

[1] A. Chorin, J. Comp. Phys., 2, 1967, pp. 12-26.

[2] A. Rizzi and L. E. Eriksson, Euler Workshop, INRIA, 1983.

[3] A. Jameson, W. Schmidt, and E. Turkel, AIAA Paper 81-1259, 1981.

[4] Y. H. Au-Yeung, Proc. Amer. Math. Soc., 20, 1969, pp. 545-548.

[5] R. Peyret, H. Viviand, Recent Advances in the Aerospace Science, 1983.

[6] E. Turkel, Euler Workshop, INRIA, 1983.

1. Report No. NASA CR-172416 ICASE Report No. 84-28	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle FAST SOLUTIONS TO THE STEADY STATE COMPRESSIBLE AND INCOMPRESSIBLE FLUID DYNAMIC EQUATIONS		5. Report Date June 1984
		6. Performing Organization Code
7. Author(s) Eli Turkel		8. Performing Organization Report No. 84-28
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665		10. Work Unit No.
		11. Contract or Grant No. NAS1-17070, NAS1-17130
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 505-31-83-01

15. Supplementary Notes

Langley Technical Monitor: R. H. Tolson
Final Report

16. Abstract

It is well known that for low speed flows the use of the compressible fluid dynamic equations is inefficient. The use of an explicit scheme requires Δt to be bounded by $1/c$. However, the physical parameters change over time scales of order $1/u$ which is much larger. Hence, it is not appropriate to use explicit schemes for very subsonic flows. Implicit schemes are hard to vectorize and frequently do not converge quickly for very subsonic flows. We shall demonstrate that if one is only interested in the steady state then a minor change to an existing code can greatly increase the efficiency of an explicit method. Even when using an implicit method the proposed changes increase the efficiency of the scheme. We shall first consider the Euler equations for low speed flows and then incompressible flows. We then indicate how to generalize the method to include viscous effects. We also show how to accelerate supersonic flow by essentially decoupling the equations.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
compressible, incompressible, subsonic, supersonic	02 - Aerodynamics 64 - Numerical Analysis Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	9	A02